

BIDDING

IN SEALED HIGH-BID AUCTIONS

A LITTLE BIT OF THEORY - Equilibrium bidding

A bidding strategy is bidding rule that applies to this auction regardless of the number on your bill. That is, the rule must tell you what to bid for any of your possible prize. Let v_i be the value of your prize. Then a bidding strategy maps each possible value into a bid $b_i = b_i(v_i)$. The higher a buyer's valuation, the more eager he is to win and so the more he will bid.

In the formal theory values are continuously distributed on some interval. Thus the probability that two buyers will have the same value is zero. It follows that the probability two buyers will make the same bid is zero. Then when theorists analyze models of bidding they can ignore the possibility of ties.

While the mathematics of bidding is subtle, it is relatively easy to analyze the DOLLAR BILL auction in which values are evenly distributed between zero and some maximum value. It is especially easy if we will consider only linear bidding strategies. That is $b_i = \alpha_i v_i$ where α_i is some strictly positive number. From our earlier argument we know that bidder i will not choose a bidding strategy with $\alpha_i > 1$.

To fix ideas, suppose that an item's value is evenly distributed over the interval $[0,100]$. Suppose that your opponent is naive and simply bids his value. That is $b_2(v_2) = v_2$. In this case you win for sure by bidding 100, you win with probability 0.75 if you bid 75, you win with probability 0.5 if you bid 50 and so on. More generally, if you bid b_1 your win probability is $b_1/100$. If you win you pay b_1 and receive v_1 so your payoff is $(v_1 - b_1)$.

Multiplying by the win probability, your expected payoff is

$$U_1(b_1) = \Pr\{b_2 < b_1\}(v_1 - b_1) = \frac{b_1}{100}(v_1 - b_1) = \frac{1}{100}(b_1 v_1 - b_1^2).$$

You can check by multiplication that your expected payoff can be rewritten as follows.

$$U_1(b_1) = \frac{1}{100}[\frac{1}{4}v_1^2 - (\frac{1}{2}v_1 - b_1)^2]$$

Note that your bid, b_1 , appears only in the quadratic term. Since this term is non-negative, the best you can do is choose your bid to make the term zero. That is choose

$$b_1 = \frac{1}{2}v_1.$$

Given this strategy, your maximum bid is 50. Thus it makes no sense for bidder 2 to bid his valuation. Suppose he chooses $\alpha_2 < 1$. If you bid b_1 you win if his bid is lower that is

if $b_2 = \alpha_2 v_2 \leq b_1$. Rearranging the inequality, you win if $v_2 < \frac{b_1}{\alpha_2}$.

As we have already noted $\Pr\{v_2 \leq 25\} = 0.25$, $\Pr\{v_2 \leq 5.0\} = 0.50$ etc. Generalizing

$$\Pr\{v_2 \leq k\} = k/100.$$

Hence

$$\Pr\{v_2 \leq \frac{b_1}{\alpha_2}\} = \frac{b_1}{100\alpha_2}.$$

This is your win probability. Thus your expected payoff is

$$\begin{aligned} U_1(b_1) &= \Pr\{b_2 < b_1\}(v_1 - b_1) = \frac{b_1}{100\alpha_2}(v_1 - b_1) \\ &= \frac{1}{100\alpha_2}[\frac{1}{4}v_1^2 - (\frac{1}{2}v_1 - b_1)^2]. \end{aligned}$$

Arguing as above it follows that your best response is still to bid half your valuation.

That is, regardless of which linear strategy your opponent choose, your best response is to bid half your valuation.

A completely symmetric argument holds for buyer 2. As long as he thinks you are using a linear strategy, his best strategy is to bid half his valuation. Then if both buyers adopt the strategy bidding half their valuations, these are mutual best responses.

John Nash (A Beautiful Mind) was the first to prove a theorem about the existence of mutual best response strategies. For this reason we usually refer to such a pair of strategies as a Nash Equilibrium.

More bidders

With more than 2 bidders we need to use calculus to solve for a bidder's best response. Suppose that bidder 2 and bidder 3 are both following linear strategies. Arguing as above, if bidder 1 bid b_1 he wins if $b_2 = \alpha_2 v_2 \leq b_1$ that is, if $v_2 \leq \frac{b_1}{\alpha_2}$. The probability of this

event is therefore $\Pr\{b_2 \leq b_1\} = \frac{b_1}{100\alpha_2}$. Similarly, for buyer 3 we have

$$\Pr\{b_3 \leq b_1\} = \frac{b_1}{100\alpha_3}.$$

Buyer 1 wins if he outbids both his opponents thus his win probability is

$$\Pr\{b_2 \leq b_1\} \times \Pr\{b_3 \leq b_1\} = \left(\frac{b_1}{100\alpha_2}\right)\left(\frac{b_1}{100\alpha_3}\right) = \frac{b_1^2}{10,000\alpha_2\alpha_3}$$

Buyer 1's expected payoff is his win probability times his payoff if he wins. That is

$$U_1(b_1) = \Pr\{b_2 \text{ and } b_3 < b_1\}(v_1 - b_1) = \frac{b_1^2}{10,000\alpha_2\alpha_3}(v_1 - b_1)$$

Ignoring the constant term, buyer 1 therefore chooses b_1 to maximize

$$R(b_1) = b_1^2(v_1 - b_1) = v_1 b_1^2 - b_1^3.$$

The derivative of R is

$$R'(b_1) = 2v_1 b_1 - 3b_1^2 = 3b_1\left(\frac{2}{3}v_1 - b_1\right).$$

Note that this is positive if $b_1 < \frac{2}{3}v_1$ and negative if $b_1 > \frac{2}{3}v_1$. Thus the best response by buyer 1 is to bid two thirds of his valuation.

Exercise: Show that with three bidders the best response is $b_1 = \frac{3}{4}v_1$. What if there are n bidders?

